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Darbouxian first integrals and invariants for real quadratic systems having an invariant conic

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Abstract
We apply the Darboux theory to study the integrability of real quadratic differential systems having an invariant conic. The fact that two intersecting straight lines or two parallel straight lines are particular cases of conics allows us to study simultaneously the integrability of quadratic systems having at least two invariant straight lines real or complex.

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1. Introduction

By definition a *real planar polynomial differential system* or simply a *polynomial system* will be a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y) \qquad \frac{dy}{dt} = \dot{y} = Q(x, y) \tag{1}$$

where the dependent variables x and y and the independent variable (the *time*) t are real and P and Q are polynomials in the variables x and y with real coefficients. Throughout this paper $m = \max\{\deg P, \deg Q\}$ will denote the *degree* of the polynomial system.

Of particular interest are the systems such that $m = 2$. The polynomial differential systems of degree 2 are called *quadratic systems* (QS). This type of equation appears in the modelization of natural phenomena described in different branches of science. One particularly well-known quadratic system is the *Lotka–Volterra system* (LV) which has been used to model the time evolution of conflicting species in biology, in chemical reactions and economics [21, 27]. Among other applications, we find a QS in astrophysics [6], in the equations of continuity describing the interactions of ions, electrons and neutral species in plasma physics (with the assumption of quasi-neutrality to eliminate either the ion or the electron equation) [25]. A reduced QS is obtained from a generalized Blasius equation for fluid flow around a wedge-shaped obstacle in boundary layer theory [11, 28]. In the context of plasma physics, all the nonlinear terms represent binary interactions or model certain transport across the boundary of the system. Moreover, the QS appears in shock waves, in neural networks, etc.

Besides their biological and physical applications, the QS have been a matter of interest for a mathematical study. More than one thousand papers have been published about QS (see [24] for a bibliographical survey). In particular, the problem of existence of first integrals has been studied by many authors, with different techniques such as Carleman embedding [3], linear compatibility analysis (developed for a QS in $\mathbb{R}^3$) [16, 26], Painlevé analysis [14, 16, 20], Lie symmetries search [1] and the old Darboux method [4, 19] based on the Darboux theorem [12].

The old Darboux method consists fundamentally in finding sufficient invariant algebraic curves in order to construct the first integrals or integrating factors. The method used in this paper is an extension of this method with the introduction of the so-called exponential factors [9, 10] and investigating the existence of the Darboux invariants, the last being a class of time-dependent first integrals.

In fact knowledge of the first integrals is of particular interest in mathematics and physics because of the possibility to have explicit expressions for the solutions of the system. However, it can be interesting sometimes to know if the system can have an invariant. Roughly speaking, with a first integral we can describe completely the phase portrait of the polynomial system, while with an invariant we only can describe its asymptotic behaviour.

In contrast with other works concerning the search for first integrals such as that of Garnier [14], we are restricted here to real QS already having an invariant algebraic conic. This particular choice helps us because the application of Darboux theory is simplified, as in general we only need to find one algebraic conic or an exponential factor more. A survey of many works related to the Darboux theory of integrability together with a first study of the integrability of real QS having an invariant conic is given in [2]. This study is completed here with the use of exponential factors and the search for Darboux invariants.

The paper is organized as follows. The main lines of Darboux theory are presented in section 2. In section 3 we obtain the nine normal forms for the real QS having one of the nine different types of conics: ellipse, complex ellipse, hyperbola, two complex straight lines intersecting at a real point, two intersecting straight lines, i.e. the Lotka–Volterra system, parabola, two parallel straight lines, two complex straight lines and one double straight line. All these are presented in sections 4–11. Finally in section 12 we give our conclusions.

2. Basic concepts and theory

We denote by $\mathbf{R}[x, y]$ or $\mathbb{C}[x, y]$, the ring of polynomials in the variables x and y with coefficients in $\mathbf{R}$ or $\mathbf{C}$, respectively.

The vector field X associated with system (1) is defined by

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}. \tag{2}$$

The polynomial system (1) is *integrable* on an open subset U of $\mathbf{R}^2$ if there exists a nonconstant analytic function $H : U \to \mathbf{R}$, called a *first integral* of the system on U, which is constant on all solution curves $(x(t), y(t))$ of system (1) on U; i.e. $H(x(t), y(t)) = \text{constant}$ for all values of t for which the solution $(x(t), y(t))$ is defined on U. Clearly, H is a first integral of system (1) on U if and only if $XH \equiv 0$ on U.

Let $R : U \to \mathbf{R}$ be an analytic function which is not identically zero on U. The function R is an *integrating factor* of the polynomial system (1) on U if one of the following three equivalent conditions holds

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y} \qquad \operatorname{div}(RP, RQ) = 0 \qquad XR = -R\operatorname{div}(P, Q) \tag{3}$$

on U. As usual the divergence of the vector field X is defined by

$$\operatorname{div}(X) = \operatorname{div}(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

The *first integral H associated with the integrating factor R* is given by the indefinite integral

$$H(x, y) = \int R(x, y)P(x, y)\,\mathrm{d}y + h(x)$$

where $h(x)$ is chosen to satisfy $\frac{\partial H}{\partial x} = -RQ$. Then

$$\dot{x} = RP = \frac{\partial H}{\partial y} \qquad \dot{y} = RQ = -\frac{\partial H}{\partial x}. \tag{4}$$

In order that this function H be well defined the open set U must be simply connected.

Conversely, given a first integral H of system (1) we always can find an integrating factor R for which (4) holds.

An invariant of system (1) on the open subset U of $\mathbf{R}^2$ is a nonconstant analytic function I in the variables x, y and t such that $I(x(t), y(t), t)$ is constant on all solution curves $(x(t), y(t))$ of system (1) contained in U.

Let $f \in \mathbf{C}[x, y]$. The algebraic curve $f(x, y) = 0$ is an *invariant algebraic curve* of the polynomial system (1) if for some polynomial $K \in \mathbf{C}[x, y]$ we have

$$Xf = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{5}$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. We note that since the polynomial system has degree m, then any cofactor has at most degree $m - 1$. Clearly, a null cofactor implies that f is a polynomial first integral.

We remark that in the definition of invariant algebraic curve $f = 0$ we always allow this curve to be complex, that is $f \in \mathbf{C}[x, y]$. As we will see, this is due to the fact that sometimes for real polynomial systems the existence of a real first integral can be forced by the existence of complex invariant algebraic curves. Of course when we look for a complex invariant algebraic curve of a real polynomial system we are thinking in the real polynomial system as a complex polynomial system.

The following result is well known (see for instance [9]).

Proposition 1. *We suppose that $f \in \mathbf{C}[x, y]$ and let $f = f_1^{n_1} \cdots f_r^{n_r}$ be its factorization in irreducible factors over $\mathbf{C}[x, y]$. Then, for a polynomial system (1), $f = 0$ is an invariant algebraic curve with cofactor K_f if and only if $f_i = 0$ is an invariant algebraic curve for each $i = 1, \ldots, r$ with cofactor K_{f_i}. Moreover, $K_f = n_1K_{f_1} + \cdots + n_rK_{f_r}$.*

Now we introduce the notion of an exponential factor described in [8] under the name ‘degenerate algebraic curve’. We will see that an exponential factor plays the role of an invariant algebraic curve when we look for a first integral of polynomial system (1).

Let $h, g \in \mathbf{C}[x, y]$ and assume that h and g are relatively prime in the ring $\mathbf{C}[x, y]$. Then the function $\exp(g/h)$ is called an *exponential factor* of a polynomial system (1) if for some polynomial $K \in \mathbf{C}[x, y]$ of degree at most $m - 1$ it satisfies the equation

$$X\left(\exp\left(\frac{g}{h}\right)\right) = K \exp\left(\frac{g}{h}\right). \tag{6}$$

As before we say that K is the *cofactor* of the exponential factor $\exp(g/h)$.

We remark that in the definition of exponential factor $\exp(g/h)$ we allow that this function be complex, that is $h, g \in \mathbf{C}[x, y]$. This is due to the same reason as in the case of invariant algebraic curves. That is, sometimes for real polynomial systems the existence of a real first integral can be forced by the existence of complex exponential factors. Again when we look for a complex exponential factor of a real polynomial system we are thinking of the real polynomial system as a complex polynomial system.

The next result is well known (see [8]).

Proposition 2. *If $F = \exp(g/h)$ is an exponential factor for a polynomial system (1) and h is not a constant, then $h = 0$ is an invariant algebraic curve, and g satisfies the equation $Xg = gK_h + hK_F$, where K_h and K_F are the cofactors of h and F, respectively.*

As far as we know, the problem of integrating a polynomial system by using its invariant algebraic curves was started by Darboux in [12]. The version that we present here improves Darboux's version essentially because here we also take into account the exponential factors (see [9]), the independent singular points (see [5]) and the invariants (see [2]).

Before stating the main results of the Darboux theory we need some definitions. If $S(x, y) = \sum_{i+j=0}^{m-1} a_{ij}x^i y^j$ is a polynomial of degree $m - 1$ with $m(m + 1)/2$ coefficients in $\mathbf{C}$, then we write $S \in \mathbf{C}_{m-1}[x, y]$. We identify the linear vector space $\mathbf{C}_{m-1}[x, y]$ with $\mathbf{C}^{m(m+1)/2}$ through the isomorphism $S \to (a_{00}, a_{10}, a_{01}, \ldots, a_{m-1,0}, a_{m-2,1}, \ldots, a_{0,m-1})$.

We say that r points $(x_k, y_k) \in \mathbf{R}^2$, $k = 1, \ldots, r$, are *independent* with respect to $\mathbf{C}_{m-1}[x, y]$ if the intersection of the r hyperplanes

$$\left\{ (a_{ij}) \in \mathbf{C}^{m(m+1)/2} : \sum_{i+j=0}^{m-1} x_k^i y_k^j a_{ij} = 0 \quad k = 1, \ldots, r \right\}$$

is a linear subspace of $\mathbf{C}^{m(m+1)/2}$ of dimension $m(m + 1)/2 - r > 0$.

We remark that the maximum number of isolated singular points of a polynomial system (1) is m^2 (by the Bezout theorem), and that the maximum number of independent isolated singular points of the system is $m(m + 1)/2$, and that $m(m + 1)/2 < m^2$ for $m \geqslant 2$.

A singular point (x_0, y_0) of system (1) is called *weak* if the divergence $\operatorname{div}(P, Q)$ of system (1) at (x_0, y_0) is zero.

The *Darboux theory* is contained in the next theorem, proved in [10] except for statement (4), which is proved here.

Theorem 3. *Suppose that a polynomial system (1) of degree m admits p invariant algebraic curves $f_i = 0$ with cofactors K_i for $i = 1, \ldots, p$, q exponential factors $\exp(g_j/h_j)$ with cofactors L_j for $j = 1, \ldots, q$, and r independent singular points $(x_k, y_k) \in \mathbf{R}^2$ such that $f_i(x_k, y_k) \neq 0$ for $i = 1, \ldots, p$ and $k = 1, \ldots, r$ and $h_j(x_k, y_k) \neq 0$ for $j = 1, \ldots, q$ and $k = 1, \ldots, r$.*

(1) *If $p+q+r < m(m+1)/2$ and the r independent singular points are weak, then substituting $f_i^{\lambda_i}$ by $|f_i|^{\lambda_i}$ if $\lambda_i \in \mathbf{R}$, the function*

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \cdots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q} \tag{7}$$

for suitable $\lambda_i, \mu_j \in \mathbf{C}$ *not all zero is either a first integral of system (1), if* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$*, or an integrating factor if* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\text{div}(P, Q)$.

(2) *If* $p + q + r = m(m+1)/2$*, and the* r *independent points are week, then there exist* $\lambda_i, \mu_j \in \mathbf{C}$ *not all zero such that* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\text{div}(P, Q)$ *and (7) is an integrating factor.*

(3) *If* $p + q + r = (m(m+1)/2) + 1$*, then there exist* $\lambda_i, \mu_j \in \mathbf{C}$ *not all zero such that* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$ *and (7) is a first integral of system (1).*

(4) *If* $p+q+r \geqslant (m(m+1)/2)+2$*, then system (1) has a rational first integral, and consequently all trajectories of the system are contained in invariant algebraic curves.*

(5) *If there exist* $\lambda_i, \mu_j \in \mathbf{C}$ *not all zero such that* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -s$ *for some* $s \in \mathbf{R}\backslash\{0\}$*, then substituting* $f_i^{\lambda_i}$ *by* $|f_i|^{\lambda_i}$ *if* $\lambda_i \in \mathbf{R}$*, the real (multi-valued) function*

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \cdots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q} \exp(st) \tag{8}$$

is an invariant of system (1).

Proof. We only prove statement (4). We denote $F_j = \exp(g_j/h_j)$. By hypothesis we have p invariant algebraic curves $f_i = 0$ with cofactors K_i, and q exponential factors F_j with cofactors L_j. That is, the f_i's satisfy $Xf_i = K_i f_i$, and the F_j's satisfy $XF_j = L_j F_j$.

We have $\lambda_i, \mu_j \in \mathbf{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -s$. Then, from

$$\begin{aligned} X & \left(f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} e^{st}\right) \\ &= \left(f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} e^{st}\right) \left(\sum_{i=1}^{p} \lambda_i \frac{Xf_i}{f_i} + \sum_{j=1}^{q} \mu_j \frac{XF_j}{F_j} + s\right) \\ &= \left(f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} e^{st}\right) \left(\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j + s\right) = 0 \end{aligned}$$

statement (4) follows. □

A (multi-valued) function of the form (7) is called a *Darbouxian* function. In particular, function (8) is called a *Darbouxian invariant*.

If among the invariant algebraic curves a complex conjugate pair $f = 0$ and $\bar{f} = 0$ occurs, then the first integral will have a factor of the form $f^{\lambda} \bar{f}^{\bar{\lambda}}$, which is just the real valued function $((\text{Re} f)^2 + (\text{Im} f)^2)^{\text{Re}\lambda} \exp(-2\,\text{Im}\,\lambda\, \arctan(\text{Im}\, f/\text{Re}\, f))$. Hence, if the polynomial system (1) is real, then the first integral obtained using the Darboux theory of integrability is also real, independently of the fact of having used complex invariant algebraic curves or complex exponential factors.

3. Real quadratic systems having an invariant conic

Quadratic systems with an invariant algebraic curve of degree 2 have been studied by many authors. For instance, Qin Yuan-xum [23] studied the quadratic systems having an ellipse as limit cycle. Druzhkova [13] formulated in terms of the coefficients of the quadratic system the necessary and sufficient conditions for existence and uniqueness of an invariant algebraic curve of second degree. A nice result is the following: a quadratic system having an algebraic invariant curve of degree 2 has at most one limit cycle. This result is essentially due to Qin Yuan-xun and Kooij and Zegeling [18] (see also Christopher [7] and Gasull [15]).

We want to study the Darbouxian integrability of real quadratic systems which have an invariant conic, i.e. an invariant algebraic curve $f(x, y) = 0$, with $f(x, y)$ a real polynomial of degree 2. For that we will look for additional invariant straight lines and exponential factors of the form $\exp(g/h)$ with degree$(g) \leqslant 3$ and with $h =$ constant, or h equal to the invariant conic or to a real factor of it (if it exists). In the special case (LV) we also look for additional invariant conics.

The conics in $\mathbf{R}^2$ are classified into ellipses (E), complex ellipses (CE), hyperbolas (H), two complex straight lines intersecting at a real point (p), two real straight lines intersecting at a point (LV), parabolas (P), two real parallel straight lines (PL), two complex parallel straight lines (CL) and one double invariant real straight line (DL). After an affine change of coordinates, we can assume that the above nine conics are given by $x^2 + y^2 - 1 = 0$, $x^2 + y^2 + 1 = 0$, $x^2 - y^2 - 1 = 0$, $x^2 + y^2 = 0$, $xy = 0$, $y - x^2 = 0$, $x^2 - 1 = 0$, $x^2 + 1 = 0$ and $x^2 = 0$, respectively.

In order to apply the Darboux theory to study the integrability of a real quadratic system having as invariant algebraic curve a conic, and simplify the computations, we start by obtaining the normal forms for such systems.

We say that a quadratic system is of type (E) if it has an invariant ellipse. In a similar way we define the quadratic systems of type (CE), (H), (p), (LV), (P), (PL), (CL) and (DL). We remark that real quadratic systems having an invariant conic of type (LV), (PL), (p) or (CL) have at least two invariant algebraic curves. Real quadratic systems having an invariant conic of type (E), (CE), (H), (P) or (DL) have, in general, only one invariant algebraic curve. The next result follows from [2].

Proposition 4. *A real quadratic system having an invariant conic after an affine change of coordinates can be written in one of the following nine forms*

$$
\begin{array}{lll}
\dot{x} = \frac{a}{2}(x^2 + y^2 - 1) + 2y(px + qy + r) & \dot{y} = \frac{b}{2}(x^2 + y^2 - 1) - 2x(px + qy + r) & (E)\\
\dot{x} = \frac{a}{2}(x^2 + y^2 + 1) + 2y(px + qy + r) & \dot{y} = \frac{b}{2}(x^2 + y^2 + 1) - 2x(px + qy + r) & (CE)\\
\dot{x} = \frac{a}{2}(x^2 - y^2 - 1) - 2y(px + qy + r) & \dot{y} = -\frac{b}{2}(x^2 - y^2 - 1) - 2x(px + qy + r) & (H)\\
\dot{x} = \frac{a}{2}(x^2 + y^2) + \frac{c}{2}x + 2y(px + qy + r) & \dot{y} = \frac{b}{2}(x^2 + y^2) + \frac{c}{2}y - 2x(px + qy + r) & (p)\\
\dot{x} = x(ax + by + c) & \dot{y} = y(Ax + By + C) & (LV)\\
\dot{x} = \frac{b}{2}xy - \frac{a}{2}(y - x^2) + px + qy + r & \dot{y} = by^2 + c(y - x^2) + 2x(px + qy + r) & (P)\\
\dot{x} = \frac{a}{2}(x^2 - 1) & \dot{y} = Q(x, y) & (PL)\\
\dot{x} = \frac{a}{2}(x^2 + 1) & \dot{y} = Q(x, y) & (CL)\\
\dot{x} = x(ax + by + c) & \dot{y} = Q(x, y) & (DL)
\end{array}
$$

if the invariant conic is of type (E), (CE), (H), (p), (LV), (P), (PL), (CL) and (DL), respectively. Here $Q(x, y)$ denotes an arbitrary polynomial of degree 2. Moreover, except for system (LV), the cofactor of the invariant conic is $ax + by + c$, where the constants b and c are zero if they do not appear in the system. To avoid reductions to linear or constant vector fields, in systems (E), (CE) and (H), we must have $(a^2 + b^2)(p^2 + q^2 + r^2) \neq 0$; in system (p) we must have $(a^2 + b^2 + c^2)(p^2 + q^2 + r^2 + c^2) \neq 0$; in system (LV) we must have $(a^2 + b^2 + A^2 + B^2)(a^2 + c^2)(B^2 + C^2) \neq 0$ and the equalities $A/a = B/b = C/c$ do not occur; in system (P) we must have $(a^2 + b^2 + c^2)(p^2 + q^2 + r^2 + b^2)(p^2 + q^2 + r^2 + a^2 + c^2) \neq 0$; in systems (PL) and (CL) we must have $a \neq 0$; and in system (DL) where the multiplicity of the invariant straight line $x = 0$ is not distinguished, we must have $a^2 + b^2 + c^2 \neq 0$.

In what follows we study the integrability of each type of quadratic system given above.

4. Quadratic systems having an invariant ellipse

Theorem 5 (Ellipse theorem)**.** *Let* $f_1 = x^2 + y^2 - 1$. *Then for a system of type (E) with* $(a^2 + b^2)(p^2 + q^2 + r^2) \neq 0$, *the following statements hold.*

(1) *If* $ap + bq = 0$ *and* $p^2 + q^2 \neq 0$, *then* $f_2 = px + qy + r = 0$ *is an invariant straight line. Moreover, if* $p^2 + b^2 \neq 0$, *then* $f_1^{-2p} f_2^{b}$ *is a first integral; and if* $a^2 + q^2 \neq 0$, *then* $f_1^{2q} f_2^{a}$ *is a first integral.*
(2) *If* $p = q = 0$, *then* $f_1^{\lambda_1}$ *is an integrating factor for* $\lambda_1 = -1$, *which gives the first integral* $H = (x^2 + y^2 - 1)^{2r} \exp(ay - bx)$.

Proof. The cofactor of $f_1 = 0$ is $K_1 = ax + by$. Since in statement (1) $ap + bq = 0$, it follows that $f_2 = 0$ is an invariant straight line of system (E) with cofactor $K_2 = -2qx + 2py$. Then, by theorem 3(1) the first part of (1) is proved. The second part of (1) follows in a similar way.

If $p = q = 0$, then $K_1 = \text{div}$. Therefore, by theorem 3(1), f_1^{-1} is an integrating factor of system (E). Hence, since $r(a^2 + b^2) \neq 0$ statement (2) follows. □

We remark that if in systems (E) we look for exponential factors of the form $\exp(g(x, y))$ or $\exp(g(x, y)/(x^2 + y^2 - 1))$ with $\text{degree}(g) \leqslant 3$, then we only find the system given in statement (2) of theorem 5.

Christopher lemma. Suppose that a polynomial system (1) of degree m has the invariant algebraic curve $f = 0$ of degree n. Let P_m, Q_m and f_n be the homogeneous components of P, Q and f of degree m and n, respectively. Then the irreducible factors of f_n must be factors of $yP_m - xQ_m$.

We note that the irreducible factors of f_n in $\mathbf{R}[x, y]$ are either linear or quadratic because f_n is a homogeneous polynomial, while the irreducible factors of f_n in $\mathbf{C}[x, y]$ are always linear. We also remark that $yP_m - xQ_m$ is the maximum degree of $\dot{\theta}$ if we write system (1) in polar coordinates $x = r\cos\theta,\ y = r\sin\theta$.

Theorem 6 (Complex ellipse theorem)**.** *For a system of type (CE) the two statements of theorem 5 hold interchanging* $x^2 + y^2 - 1$ *with* $x^2 + y^2 + 1$.

Proof. The same proof as in Theorem 5. □

5. Quadratic systems having an invariant hyperbola

Theorem 7 (Hyperbola theorem)**.** *Let* $f_1 = x^2 - y^2 - 1$. *Then for a system of type (H) with* $(a^2 + b^2)(p^2 + q^2 + r^2) \neq 0$, *the statements given in Table 1 hold.*

If system (H) has an invariant straight line $f_2 = 0$, *then it satisfies one of the statements (1)–(3). If system (H) has an exponential factor* f_2 *of the form* $\exp(g(x, y))$ *or* $\exp(g(x, y)/(x^2 - y^2 - 1)$ *with* $\text{degree}(g) \leqslant 3$, *then it satisfies one of statements (4)–(9).*

Proof. The cofactors of $f_1 = 0$ and $f_2 = px + qy + r = 0$ are $K_1 = ax + by$ and $K_2 = -2(qx + py)$, respectively. So, the first two statements can be proved as in Theorem 5. We prove the first option of statement (3). Assume that $a + b = r = 0$. Then it is easy to check that $f_2 = 0$ is an invariant straight line of system (H) with cofactor $K_2 = 2(px + qy)$. The divergence of (H) is $\text{div} = (a - 2q)x + (b - 2p)y$. Hence the equation $\lambda_1 K_1 + \lambda_2 K_2 = -\text{div}$, has the solution $\lambda_1 = (2q - 2p - a)/a$, $\lambda_2 = 1$. Therefore, by theorem 3(1), we obtain (3). Similarly proved are statements (4)–(6), where the exponential factors have respectively

Table 1. Conditions for existence of invariant lines, exponential factors, first integrals, invariants or integrating factors for the hyperbola system.

Statement	Conditions	f_2	H or R
(1)	$ap - bq = 0, (p^2+q^2)(a^2+q^2) \neq 0$	$px + qy + r = 0$	$H = f_1^{2q} f_2^a$
	$ap - bq = 0, (p^2+q^2)(b^2+p^2) \neq 0$	$px + qy + r = 0$	$H = f_1^{2p} f_2^b$
(2)	$p = q = 0$		$R = 1/f_1$
(3)	$a \pm b = r = 0, p \mp q \neq 0$	$x \mp y = 0$	$R = f_1^{(2q \mp 2p - a)/a} f_2$
(4)	$\pm a + b + 4p = p \mp q = r = 0$	$\exp(y \mp x)$	$H = f_2^{1/(2p)} \mathrm{e}^t$
(5)	$a = p = r = b \mp 2q = 0$	$\exp((x \mp y)/f_1)$	$H = f_2 \mathrm{e}^{-qt}$
(6)	$a \mp 4p = b \mp 4q = r = 0, a \pm b \neq 0$	$\exp((x \mp y)^2/f_1)$	$R = f_1^{-1/2} f_2^{2(q \mp p)/(a \pm b)}$
(7)	$a + 4q = b \mp 2q = p = r = 0$	$\exp((x((x \mp y)^2 - 1) \pm 2y)/f_1)$	$H = f_2$
(8)	$a \mp 2p = b + 4p = q = r = 0$	$\exp((y((x \mp y)^2 + 1) \mp 2x)/f_1)$	$H = f_2$
(9)	$a = r = b + 3p = p \mp 2q = 0$	$\exp((x^3 - 3xy^2 \pm 2y^3 - 9x)/f_1)$	$H = f_2$

$-2p$, q and $4(q \pm p)(x \mp y)$ as cofactors. In statements (7)–(9), $K_2 = 0$, i.e. the exponential factor is a first integral.

The following first integals have been computed from the integrating factors of statements (2) and (3):

(2)$H = f_1^{2r} \exp(bx + ay)$.

(3)$H = f_1^{2(q \mp p)}((p \mp q)((a \mp 4p)x^2 + 2(2p \mp 2q \mp a)xy + (a + 4q)y^2) + a(p \pm q))^a$. □

6. Quadratic systems having two invariant complex straight lines intersecting at a real point

The next theorem characterizes all real quadratic systems of type (p) which are integrable having at least three invariant curves of degree 1. Of course, systems (p) always have two complex straight lines $x + \mathrm{i}y = 0$ and $x - \mathrm{i}y = 0$.

Theorem 8 (Two invariant complex straight lines intersecting at a real point theorem)**.** *We assume that* $(a^2 + b^2 + c^2)(p^2 + q^2 + r^2 + c^2) \neq 0$.

Let $f_1 = x + \mathrm{i}y$ *and* $f_2 = \bar{f}_1$. *If a system (p) has a third invariant algebraic curve* $f_3 = 0$ *of degree 1, then it satisfies one of the statements (1)–(7) given in Table 2.*

If a system (p) has an exponential factor of the form $\exp(g(x, y))$ *or* $\exp(g(x, y)/(x^2+y^2))$ *with degree*$(g) \leqslant 3$, *then it satisfies statements (8)–(11) given in Table 2.*

Proof. The singular points of system (p) are computed using the resultant of the polynomials P and Q with respect to the variable y, which turns out to be a polynomial of the form $xT(x) = x(A + Bx + Cx^2 + Dx^3)$. By the properties of the resultant, we know that if (x_0, y_0) is a singular point of system (p), then x_0 is a root of the resultant. We define

$$\omega = 27A^2D^2 + 2(2B^2 - 9AC)BD + (4AC - B^2)C^2.$$

Then ω is equal to

$$64(cp + 4rq)^2(b^2 + (a + 4q)^2)^2(16(cq - br)^2 + 16(cp - ar)^2 + c^2(a^2 + b^2 + 8aq - 8bp))^2.$$

It is well known for a polynomial $T(x)$ of degree 3 ($D \neq 0$) that $T(x)$ has a unique simple real root if $\omega > 0$, one simple real root and one double root or a triple real root if $\omega = 0$, and three simple real roots if $\omega < 0$.

Table 2. Conditions for existence of invariant lines, exponential factors, first integrals, invariants or integrating factors for the (p) system ($v = b - 4p$, $w = a + 4q$).

Statement	Conditions	f_3, H or R
(1)	$wc^2 + 16r(ar - cp) = 0$	
	$vc^2 - 16r(cq - br) = 0$	$H = (x^2 + y^2)^{2r} \exp\left(c \arctan\left(\frac{y}{x}\right)\right)$
(2)	$p(4ar + cv) + q(4br - cw) = 0$	$f_3 = vx - wy - 4r + (4br - cw)/(4p) = 0$
	$v^2 + w^2 \neq 0$	$H = f_1^{\lambda_1} \bar{f}_1^{\bar{\lambda}_1} f_3^{\lambda_3}$ [a]
(3)	$r = 0, c(v^2 + w^2) \neq 0$	$f_3 = vx - wy = 0$
		$R = f_1^{\lambda_1} \bar{f}_1^{\bar{\lambda}_1} f_3^{\lambda_3}$, $H = f_1^{\mu_1} \bar{f}_1^{\bar{\mu}_1} f_3^{\mu_3} e^t$ [b]
(4)	$c = v = w = 0$	$f_3 = px + qy + r = 0$
		$H = (px + qy + r)^2/(x^2 + y^2)$
(5)	$a + 2q = b - 2p = c = r = 0$	$f_3 = bx - ay = 0$, $H = (bx - ay)/(x^2 + y^2)$
(6)	$a = c = q = b - 2p = 0$	$f_3 = bx + 2r = 0$, $H = (bx + 2r)/(x^2 + y^2)$
(7)	$a - 2q = b = c = p = 0$	$f_3 = ay + 2r = 0$, $H = (x^2 + y^2)(ay + 2r)$
(8)	$a = p = q = 0$	$f_3 = \exp(x)$
	$a = p = q = c = 0$	$H = (x^2 + y^2)^{-2r} f_3^b$
(9)	$b = c = p = q = 0$	$f_3 = \exp(y)$, $H = (x^2 + y^2)^{2r} f_3^a$
(10)	$v = w = r = 0$	$f_3 = \exp(xy/(x^2 + y^2))$, $H = xy/(x^2 + y^2)$
(11)	$a = c = q = b - 6p = 0$	$f_3 = H = \exp((b^3x^3 - 18b^2ry^2 + 108br^2x + 216r^3)/(x^2 + y^2))$

[a] For $\lambda_1 = (cp + 4rq)(8r - 2\mathrm{i}c)$, $\lambda_3 = a(c^2 + 16r^2) + 4c(cq - 4rp)$. If moreover $(c^2 + r^2)(a^2 + c^2 + q^2)(a^2 + p^2 + q^2)(b^2 + c^2 + p^2) \neq 0$, a real expression of the first integral is $H = (x^2 + y^2)^{8r(cp+4rq)} \exp(4c(cp + 4rq) \arctan(y/x))(vx - wy - 4r + (4br - cw)/(4p))^{a(c^2+16r^2)+4c(cq-4rp)}$.

[b] For $\lambda_1 = -(a^2 + b^2 + 6(aq - bp) + 8(p^2 + q^2) + 2\mathrm{i}(bq + ap))/(v^2 + w^2)$, $\lambda_3 = 4(pv - qw)/(v^2 + w^2)$, $\mu_1 = 4(pv - qw + \mathrm{i}(bq + ap))/(c(v^2 + w^2))$, $\mu_3 = -2(bv + aw)/(c(v^2 + w^2))$.

If $(a + 4q)c^2 + 16r(ar - cp) = 0$ and $(b - 4p)c^2 - 16r(cq - br) = 0$, then $\omega = 0$ and working with the roots of $T(x)$, the resultant has three real roots, and one of them is double. Otherwise $\omega > 0$ and the resultant has at most two real roots, or it is identically zero. Then system (p) has three real singular points, one of them being the origin. The other two are not contained in the algebraic curves $x + \mathrm{i}y = 0$ and $x - \mathrm{i}y = 0$. So, by Theorem 3(1) it follows that $H = f_1^{\lambda_1} \bar{f}_1^{\bar{\lambda}_1}$ is a first integral of (p) with $\lambda_1 K_1 + \bar{\lambda}_1 \bar{K}_1 = 0$, K_1 being the cofactor of f_1 which is $K_1 = \left(\frac{a}{2} + \mathrm{i}\left(\frac{b}{2} - 2p\right)\right) x + \left(\frac{b}{2} - \mathrm{i}\left(\frac{a}{2} + 2q\right)\right) y + \frac{c}{2} - 2\mathrm{i}r$. So, statement (1) is proved.

With the assumptions of statement (2), the cofactor of f_3 is $K_3 = 2(-qx + py)$. Since $p(4ar - c(4p - b)) + q(4br - c(4q + a)) = 0$, it follows that $\lambda_1 K_1 + \bar{\lambda}_1 \bar{K}_1 + \lambda_3 K_3 = 0$ for $\lambda_1 = (cp + 4rq)(8r - 2\mathrm{i}c)$ and $\lambda_3 = a(c^2 + 16r^2) + 4c(cq - 4rp)$. Therefore, by Theorem 3(1) we obtain that $f_1^{\lambda_1} \bar{f}_1^{\bar{\lambda}_1} f_3^{\lambda_3}$ is a first integral, and consequently statement (2) is proved. Now we suppose the hypothesis of statement (3). Then the cofactor of f_3 is $K_3 = -2qx + 2py + c/2$. Since $c((4p - b)^2 + (4q + a)^2) \neq 0$, the system $\lambda_1 K_1 + \bar{\lambda}_1 \bar{K}_1 + \lambda_3 K_3 = (2q - a)x - (2p + b)y - c$ has a unique solution for λ_1 and λ_3, the one described in the theorem. Then, by Theorem 3(1) $f_1^{\lambda_1} \bar{f}_1^{\bar{\lambda}_1} f_3^{\lambda_3}$ is an integrating factor of (p). Since the system $\mu_1 K_1 + \bar{\mu}_1 \bar{K}_1 + \mu_3 K_3 = -s$ has a unique solution for μ_1 and μ_3, the one described in the theorem, by Theorem 3(5), $f_1^{\mu_1} \bar{f}_1^{\bar{\mu}_1} f_3^{\mu_3} e^{st}$ is an invariant of (p). Hence, statement (3) is proved. Similarly, statements (4)–(7), which have $K_3 = -2qx + 2py$ as cofactor, and statements (8) and (9) with, respectively, $cx/2 + 2ry$ and $-2rx$ as cofactor for f_3 are proved. In statements (10) and (11), $K_3 = 0$, i.e. the exponential factor is a first integral.

To check that statements (2)–(7) of the theorem give all invariant algebraic curves of systems (p) of degree 1, we must find all polynomials f_3 of degree 1 satisfying (5). This tedious task can be made easier with the help of an algebraic manipulator and the Christopher lemma, which fix the form of f_3 to be $f_3 = (4p - b)x + (4q + a)y +$ constant. □

7. Quadratic systems having two invariant real straight lines intersecting at a point

The systems (LV) are called Lotka–Volterra systems. Their study was started by Lotka and Volterra in [21, 22, 27]. Later on Kolmogorov studied them in [17] and some authors call the systems (LV) *Kolmogorov systems*. The next theorem characterizes all Lotka–Volterra systems which are integrable having at least three invariant algebraic curves of degree 1 or an exponential factor of the form $\exp(g(x, y)/h(x, y))$ with degree $(g) \leqslant 3$, degree $(h) \leqslant 2$. Of course, Lotka–Volterra systems in the normal form (LV) always have two invariant straight lines $x = 0$ and $y = 0$.

Theorem 9 (Two invariant real straight lines intersecting at a point theorem)**.** *$f_1 = x$ and $f_2 = y$ are invariant straight lines of a Lotka–Volterra system (LV). If $(a^2 + b^2 + A^2 + B^2) \times (a^2+c^2)(B^2+C^2) \neq 0$ and $A/a = B/b = C/c$ is not satisfied, then the statements of Table 3 hold, modulo the symmetry $(x, y, a, b, c, A, B, C) \to (y, x, B, A, C, b, a, c)$.*

Table 3. Conditions for existence of invariant lines, exponential factors, first integrals, invariants or integrating factors for the Lotka–Volterra system.

Statement	Conditions	f_3, H or R
(1)	$aB - bA = 0, bC - Bc \neq 0$	$H = x^{B/(bC-Bc)} y^{-b/(bC-Bc)} e^t$
(2)	$b = 0,\ ac \neq 0$	$f_3 = ax + c = 0$, $R = x^{(C-c)/c} y^{-2} f_3^{(cA-ac-aC)/(ac)}$
		$H = x^{-1}(ax + c)\, e^{ct}$
(3)	$c - C = 0, (A - a)(B - b) \neq 0$	$f_3 = (A - a)x + (B - b)y = 0$
		$R = x^{b/(B-b)} y^{A/(a-A)} f_3^{(ab-2aB+AB)/((a-A)(B-b))}$
		$H = x^{B/(b-B)} y^{a/(A-a)} f_3^{(aB-bA)/((a-A)(B-b))} e^{Ct}$
(4)	$r_{12} = cB(A - a) + aC(b - B) = 0,$	$f_3 = aCx + cBy + cC = 0$
	$acBC(a - A)(b - B) \neq 0$	$H = x^{(A-a)B} y^{a(b-B)} f_3^{aB-bA}$
(5)	$a - A = b - B = 0$	$f_3 = ACx + cBy + cC = 0$
	$a - A = b - B = 0$ and $c - C \neq 0$	$H = x^C y^{-c} f_3^{c-C}$
(6)	$a = c = 0, bB \neq 0$	$f_3 = ABx + (B - b)(By + C) = 0$, $H = x^{-B} f_3^b$
(7)	$a = b = 0, c \neq 0$	$f_3 = \exp(x)$, $R = x^{(C-c)/c} y^{-2} f_3^{A/c}$, $H = x^{-1} e^{ct}$
(8)	$a = B = 0$	$f_3 = \exp(Ax - by)$, $H = x^C y^{-c} f_3$
(9)	$b = c = 0, a \neq 0$	$f_3 = \exp(1/x)$, $R = x^{(A-2a)/a} y^{-2} f_3^{-C/a}$, $H = 1/x + at$
(10)	$c - C = b - B = 0, a - A \neq 0$	$f_3 = \exp(y/x)$, $R = x^{(2a-A)/(A-a)} y^{A/(a-A)} f_3^{B/(a-A)}$
	$c - C = b - B = 0, C \neq 0$	$f_3 = \exp(y/x)$, $H = x^A y^{-a} f_3^{-B} e^{(a-A)Ct}$
(11)	$b - B = c = 0$	$f_3 = \exp((By + C)/x)$, $H = x^{-A} y^a f_3$

If system (LV) has a third invariant straight line, it satisfies one of the statements (2)–(6). If system (LV) has an exponential factor of the form $\exp(g(x, y))$, $\exp(g(x, y)/x)$, $\exp(g(x, y)/(xy))$ *or* $\exp(g(x, y)/x^2)$ *with degree$(g) \leqslant 3$, then it satisfies statements (7)–(11).*

Proof. As the cofactors of $x = 0$ and $y = 0$ are, respectively, $K_1 = ax + by + c$ and $K_2 = Ax + By + C$, under the assumptions of statement (1) the equation $\mu_1 K_1 + \mu_2 K_2 + s = 0$ has the solution in statement (1).

Under the assumptions of statement (2) the cofactor of f_3 is $K_3 = ax$. Since $ac \neq 0$, the system generated from the equation $\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3 + (2a + A)x + 2By + c + C = 0$ has a unique solution $\lambda_1 = (C - c)/c, \lambda_2 = -2, \lambda_3 = (cA - ac - aC)/(ac)$. Therefore, by theorem 3(1) we obtain that $x^{\lambda_1} y^{\lambda_2} f_3^{\lambda_3}$ is an integrating factor of system (LV). The system generated from the equation $\mu_1 K_1 + \mu_2 K_2 + \mu_3 K_3 = -s$ has a unique solution

$\mu_1 = -s/c$, $\mu_2 = 0$, $\mu_3 = s/c$. Hence, by theorem 3(5), $x^{\mu_1} y^{\mu_2} f_3^{\mu_3} e^{st}$ is an invariant of (LV). So, statement (2) is proved.

Now we assume the hypotheses of statement (3). Then the cofactor of f_3 is $K_3 = ax + By + c$. Since $(A - a)(B - b) \neq 0$, then from $\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3 + (2a + A)x + (b + 2B)y + 2c = 0$ we obtain the unique solution $\lambda_1 = b/(B - b)$, $\lambda_2 = A/(a - A)$, $\lambda_3 = (ab - 2aB + AB)/((a - A)(B - b))$. Then, by Theorem 3(1) we have that $x^{\lambda_1} y^{\lambda_2} f_3^{\lambda_3}$ is an integrating factor of (LV). The system $\mu_1 K_1 + \mu_2 K_2 + \mu_3 K_3 = -s$ has the unique solution $\mu_1 = sB/(C(b-B))$, $\mu_2 = as/(C(A-a))$, $\mu_3 = (aB-bA)s/(C(a-A)(B-b))$. Therefore, from Theorem 3(5) it follows that $x^{\mu_1} y^{\mu_2} f_3^{\mu_3} e^{st}$ is an invariant of (LV).

Suppose that $r_{12} = 0$ and $acBC(a - A)(b - B) \neq 0$, then $f_3 = aCx + cBy + cC = 0$ is an invariant straight line of (LV) with $K_3 = ax + By$ and a solution of $\sum_{i=1}^{3} \lambda_i K_i = 0$ is $\lambda_1 = (A - a)B$, $\lambda_2 = a(b - B)$, $\lambda_3 = aB - bA$. Hence, by Theorem 3(1), we obtain (4).

Similarly, statements (5) and (6), where the respective invariant straight lines f_3 have $Ax + By$ and By as cofactors and statements (7)–(11), where the respective exponential factors have cx, $Acx - bCy$, $-a$, $(A - a)y$, $B(A - a)y - aC$ as cofactors are proved.

Finally, in order to check that modulo the symmetry $(x, y, a, b, c, A, B, C) \rightarrow (y, x, B, A, C, b, a, c)$, statements (2)–(6) of the theorem take into account all invariant algebraic curves of the Lotka–Volterra system (LV) of degree 1, we must find all polynomials f_3 of degree 1 satisfying equation (5). This tedious task is done with the help of the algebraic manipulator Maple, and the Christopher lemma. We note that the Christopher lemma says that f_3 must be of the form $f_3 = (a - A)x + (b - B)y +$ constant, $f_3 = x +$ constant or $f_3 = y +$ constant. Moreover, due to the mentioned symmetry it is sufficient to consider the first two possibilities. □

We can compute the first integral associated with the integrating factor of statement (2) of Theorem 9, it is

$$H = \frac{x^{C/c}}{Cy} \left(C(c + ax)^{(cA - aC)/(ac)} + Bc^{(cA-aC)/(ac)} y F\left(\frac{C}{c}, \frac{ac - cA + aC}{ac}, 1 + \frac{C}{c}; -\frac{ax}{c} \right) \right)$$

where $F(a, b; c; x)$ is the hypergeometric function

$$F(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!} .$$

Here we have used the notation

$$(a)_k = \begin{cases} 1 & \text{if} \quad k = 0 \\ a(a + 1)(a + 2) \cdots (a + k - 1) & \text{if} \quad k > 0. \end{cases}$$

The hypergeometric function $F(a, b; c; x)$ is a solution of the hypergeometric differential equation

$$x(1 - x)y'' + (c - (a + b + 1)x)y' - aby = 0 .$$

Note that statement 9(8) with $A = -b$ was found and generalized to the n-dimensionsal LV by Volterra [27]. Moreover, Cairó and Feix [3] found the invariants of statements 9(1) and 9(3) and the first integral 9(4) and also generalized them to the n-dimensional Lotka–Volterra system.

The next theorem characterizes all Lotka–Volterra systems which are integrable having an invariant conic different from $xy = 0$ or an exponential factor $\exp(g(x, y)/h(x, y))$ with degree$(g) \leqslant 3$ and h equal to $1, x, xy$ or x^2.

Theorem 10. *Let $f_1 = x$ and $f_2 = y$. Assume that $(a^2 + b^2 + A^2 + B^2)(a^2 + c^2)(B^2 + C^2) \neq 0$ and that $A/a = B/b = C/c$ do not hold simultaneously, then modulo the symmetry*

Table 4. Conditions for existence of invariant conics, exponential factors, first integrals, invariants or integrating factors for the Lotka–Volterra system.

Statement	Conditions	f_3, H or R
(1)	$B-2b=a(2c+C)-cA=0$, $cC\neq 0$	$f_3=C(ax+c)^2+c^2By=0$ $R=x^{-2}y^{(c-C)/C}f_3^{-(2c+C)/(2C)}$, $H=x^{-2}f_3\,\mathrm{e}^{2ct}$
(2)	$b+B=a(2c+C)-cA=0$, $ac(c+C)\neq 0$	$f_3=(c+C)(ax+c)^2+2acBxy=0$ $R=x^{-c/(c+C)}y^{-(2c+C)/(c+C)}f_3^{(c-C)/(2(c+C))}$ $H=x^{-1}y^{-1}f_3^{(3c+C)/(2c)}\,\mathrm{e}^{(c+C)t}$
(3)	$B(c+2C)-bC=a(2c+C)-cA=0$, $cC\neq 0$	$f_3=(aCx+cBy+cC)^2-4acBCxy=0$ $R=x^{-1}y^{-1}f_3^{-1/2}$, $H=x^{-C}y^{-c}f_3^{c+C}\,\mathrm{e}^{2cCt}$
(4)	$a(B-b)+(b-2B)(A-a)=0$, $c-2C=0$, $2B-b\neq 0$	$f_3=acx+(ax+(2B-b)y)^2=0$ $R=x^{(B-b)/(b-2B)}y^{-2}f_3^{b/(2(b-2B))}$ $H=x^By^{b-2B}f_3^{B-b}\,\mathrm{e}^{C(b-2B)t}$
(5)	$a-2A=b-2B=c=0$, $C\neq 0$	$f_3=\exp((By+C)^2/x)$ $R=x^{-3/2}y^{-2}f_3^{-1/(2AC)}$, $H=x^{1/2}y^{-1}\,\mathrm{e}^{Ct}$
(6)	$a=b-2B=c-2C=0$, $A\neq 0$	$f_3=\exp((Ax-By)^2/x)$ $R=x^{-1/2}y^{-2}f_3^{1/(2AC)}$, $H=xy^{-1}f_3^{1/(2AC)}\,\mathrm{e}^{-Ct}$
(7)	$a-A=b+B=c+C=0$, $abAB\neq 0$	$f_3=\exp((C-Ax)^2/(xy))$ $R=x^{-1/2}y^{-3/2}f_3^{-1/(2ab)}$, $H=xyf_3^{-1/(ab)}\,\mathrm{e}^{-2Ct}$

$(x, y, a, b, c, A, B, C) \to (y, x, B, A, C, b, a, c)$*, the statements of Table 4 hold. If the (LV) has a third invariant algebraic curve* $f_3 = 0$ *of degree 2, it satisfies one of the statements (1)–(4). If the (LV) has an exponential factor of the form* $\exp(g(x, y))$, $\exp(g(x, y)/x)$, $\exp(g(x, y)/(xy))$ *or* $\exp(g(x, y)/x^2)$ *with degree*$(g) \leqslant 3$*, then it satisfies statements (5)–(7).*

Proof. In statements (1)–(4), the K_3 are, respectively, $2(ax + by)$, $2ax$, $2(ax + b(A - 2a)/(2A - 3a))$ and $2(ax+b(A-2a)/(2A-3a)+C)$. Then, according to Theorem 3(1) $x^{\lambda_1}y^{\lambda_2}f_3^{\lambda_3}$ is an integrating factor, if and only if $\sum_{i=1}^{3}\lambda_i K_i+\mathrm{div}=0$. Again, to see that $x^{\mu_1}y^{\mu_2}f_3^{\mu_3}\mathrm{e}^{st}$ is an invariant of system (LV) in statements (1)–(4), it is sufficient to verify that $\sum_{i=1}^{3}\mu_i K_i+s=0$ from Theorem 3(5). When conditions $a = 2A$, $b = 2B$ and $c = 0$ are satisfied, then $f_3 = \exp((By + C)^2/x)$ is an exponential factor with cofactor $K_3 = -2AC(By + C)$ and by statements (1) and (5) of Theorem 3 we obtain (5). Similarly proved are statements (6) and (7), where the respective exponential factors have $2AC(Ax - By)$ and $2ab(Ax - C)$ as cofactors.

To check that modulo the symmetry $(x, y, a, b, c, A, B, C) \to (y, x, B, A, C, b, a, c)$, statements (1)–(5) of the theorem give all invariant algebraic curves of the Lotka–Volterra system (LV) of degree 2, we must find all polynomials f_3 of degree 2 satisfying equation (5). We note that the Christopher lemma and the mentioned symmetry imply that f_3 must be of the form $f_3 = f_{00} + f_{10}x + f_{01}y + x^2$, $f_3 = f_{00} + f_{10}x + f_{01}y + xy$, $f_3 = f_{00} + f_{10}x + f_{01}y + x((a - A)x + (b - B)y)$ or $f_3 = f_{00} + f_{10}x + f_{01}y + ((a - A)x + (b - B)y)^2$. □

Note that the statements of theorems 9(5), 9(7) and 10(5) satisfy the condition $aB - bA = 0$ of theorem 9(1), but they are not particular cases of it because they concern first integrals or integrating factors. Actually the condition $aB - bA = 0$ alone gives only an invariant. Moreover, exclusion conditions for statements of theorems 9(2), 9(3), 9(4), 10(2) and 10(4) are found in the statements of theorems 9(7) and 9(9) for 9(2), 9(10) for 9(3), 9(5), 9(6), 9(8) and 9(11) for 9(4), 10(7) for 10(2) and 10(6) for 10(4). The exclusion conditions

$b = B = 0$ and $c = C = 0$ of the statement of theorem 9(4) are particular cases of the statements of theorems 9(2) and 9(3), respectively. Finally the excluded cases of 10(1) and 10(2), namely $cC \neq 0$, are particular cases of 9(3).

8. Quadratic systems having an invariant parabola

In the next theorem, first we study the Darboux integrability in system (P) using only the parabola $y - x^2 = 0$, thus we obtain statements (1), (2) and (6). After, we study the Darboux integrability of system (P) with only the parabola and the existence of a unique invariant straight line. We note that the integrability forced by the parabola and the existence of two real invariant straight lines which intersect at a point has been analysed in Theorem 10. There are cases of integrability forced by the existence of a parabola and two parallel invariant straight lines, as the system $b = 0$, $a - 2q = 0$, $q \neq 0$ and $A^2 = p^2 - 4qr > 0$. Then $f_2 = 2qx + p + |A| = 0$ and $f_3 = 2qx + p - |A| = 0$ are two parallel invariant straight lines, and $H = f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ is a first integral for $\lambda_1 = |A|$, $\lambda_2 = c - p - |A|$, $\lambda_3 = p - c - |A|$. However, here we do not study these cases. We also study the integrability forced by the parabola and the existence of an exponential factor of the form $\exp(g(x, y))$ or $\exp(g(x, y)/(y - x^2))$ with degree$(g) \leqslant 3$.

Theorem 11 (Parabola theorem). *Let $f_1 = y - x^2$. Then for a system of type (P) with $(a^2 + b^2 + c^2)(p^2 + q^2 + r^2 + b^2)(p^2 + q^2 + r^2 + a^2 + c^2) \neq 0$, the following statements hold.*

(1) *If $b = 0$, $ap - 2cq = 0$ and $aq \neq 0$, then $f_1^{\lambda_1}$ is an integrating factor for $\lambda_1 = -(a+2q)/a$ which gives the first integral $H = (2cqx + aqy + ra)^a/(y - x^2)^{2q}$.*

(2) *If $3a - 4q = 0$ and $ap - 2cq = 0$, then $f_1^{\lambda_1}$ is an integrating factor for $\lambda_1 = -5/2$ which gives the first integral $H = (-4bx^3 + 6bxy + 6cx + 3ay + 4r)/(y - x^2)^{3/2}$.*

(3) *If $a = b = 0$, then $f_1 e^{-ct}$ is an invariant.*

(4) *If $A^2 = a^2 - 8aq + 16q^2 + 8bc - 16bp > 0$, $c_1 = 0$, $c_2 = 0$ (see the definitions of c_1 and c_2 in the appendix) and $b(3a - 4q - |A|) \neq 0$, then $f_2 = (4q - a - |A|)x + 2by + 2f_{00} = 0$ where f_{00} satisfies $b(3(a + A) - 4q)f_{00} + (-a^2 + 6aq + 2b(p - c) - 8q^2)|A| - a^3 + 32q^2(q - a) + 10a(aq + bp) + 8b(2c - 3p - br) = 0$ is an invariant straight line, and $f_1^{\lambda_1} f_2^{\lambda_2}$ is an integrating factor for $\lambda_1 = (4q - 3a - 5|A|)/(2(3a - 4q - |A|))$, $\lambda_2 = 2(3a - 4q)/(3a - 4q - |A|)$.*

(5) *If $2b^2r + (a - 2q)(a^2 + 2bp - 2aq) = 0$, $cb - (a^2 + 2bp - 2aq) = 0$ and $b \neq 0$, then $f_2 = bx + 2q - a = 0$ is an invariant straight line, and $H = f_1^{\lambda_1} f_2^{\lambda_2}$ is a first integral for $\lambda_1 = 1$, $\lambda_2 = -2$.*

(6) *If $b = 0$, $a - 2q = 0$, $p^2 - 4qr = 0$ and $q \neq 0$, then the straight line $f_2 = 2qx + p = 0$ is an invariant straight line of (P) and $f_1^{\lambda_1} f_2^{\lambda_2}$ is a Darboux integrating factor for $\lambda_1 = -1$, $\lambda_2 = -2$, which gives the first integral $H = \frac{2(c-p)}{2qx+p} + \ln\left|\frac{4q(y-x^2)}{(2qx+p)^2}\right|$.*

(7) *If $a = b = q = 0$ and $p \neq 0$, then $f_2 = px + r = 0$ is an invariant straight line, and $f_1^{\lambda_1} f_2^{\lambda_2}$ is an integrating factor for $\lambda_1 = \lambda_2 = -1$, which gives the first integral $H = (y - x^2)^p/(px + r)^c$.*

(8) *If $64b^2r - a(4q - a)^2 = 0$, $32bp + (a - 4q)(3a + 4q) = 0$, $16bc + (a - 4q)(5a - 4q) = 0$ and $b \neq 0$, then $f_2 = (4q - a)x + 2by + (4q - a)^2/(8b) = 0$ is an invariant straight line, and $f_1^{\lambda_1} f_2^{\lambda_2}$ is an integrating factor for $\lambda_1 = -1/2$, $\lambda_2 = -2$, which gives the first integral*

$$H = \arctan\left(\frac{4b\sqrt{y - x^2}}{4bx + 4q - a}\right) + \frac{(3a - 4q)\sqrt{y - x^2}}{2((4q - a)x + 2by + (4q - a)^2/(8b))}.$$

(9) *If* $3bp - 2q^2 = 0$, $27b^2r - 4q^3 = 0$, $9bc + 4q^2 - 6aq = 0$ *and* $bq \neq 0$, *then* $f_2 = 12qx + 9by + 4q^2/b = 0$ *is an invariant straight line, and* $f_1^{\lambda_1} f_2^{\lambda_2}$ *is an integrating factor for* $\lambda_1 = -1$, $\lambda_2 = -3/2$, *which gives the first integral*

$$H = \frac{6a - 8q}{\sqrt{bf_2}} - \ln \left| \frac{9b^2x^2 + (24bq + 6b\sqrt{bf_2})x + 9b^2y + 8q^2 + 4q\sqrt{bf_2}}{9b^2(x^2 - y)} \right|.$$

If system (P) has an exponential factor of the form $\exp(g(x, y))$ *or* $\exp(g(x, y)/(y - x^2))$, *with* $0 \leqslant$ *degree*$(g) \leqslant 3$, *then it satisfies one of the following statements.*

(10) *If* $r(a - 2q) + c(p - c) = 0$, $bc + 2(a - q)(a - 2q) = 0$ *and* $((a - q)^2 + r^2 + c^2)((a - 2q)^2 + c^2) \neq 0$, *then* $f_2 = \exp((cx + (a - q)y + r)/(y - x^2))$ *is an exponential factor and* $H = 2(cx + (a - q)y + r)/(y - x^2) + (c(2q - 3a) + 4p(a - q))t$ *is an invariant.*

(11) *If* $bc + 2(a - q)(a - 2q) = 0$, $pc - (2rq + c^2 - ra) = 0$, *and* $c \neq 0$, *then* $f_2 = \exp((cx + (a - q)y + r)/(y - x^2))$ *is an exponential factor. If* $a \neq 0$ *and* $a - q = 0$ *then* $f_1^{\lambda_1} f_2^{\lambda_2}$ *is an integrating factor for* $\lambda_1 = -3$ *and* $\lambda_2 = 2(c^2 - ar)/(ac^2)$ *which gives the first integral* $H = (2(ar - c^2)(ax + c) + a^2c(y - x^2))/(y - x^2)\exp(-2(ar - c^2)(cx + r)/(ac^2(y - x^2)))$.

(12) *If* $3a - 4q = 0$, $4b(2br - ap) + a^3 = 0$, $4bc - a^2 = 0$ *and* $acq \neq 0$, *then* $f_2 = \exp((cx + ay/4 + r)/(y - x^2))$ *is an exponential factor, and* $f_1^{\lambda_1} f_2^{\lambda_2}$ *is an integrating factor for* $\lambda_1 = -5/2$ *and* $\lambda_2 = 2/a$ *which gives the first integral* $H = (ax + 2c)^{2a}(y - x^2)^{-a}\exp((4cx + ay + 4r)/(y - x^2))$.

(13) *If* $b = q = 0$, *then* $f_2 = \exp(2(c - 2p)x + ay)$ *is an exponential factor. Moreover, if* $p = 0$, *then* $H = (y - x^2)^{-2r}\exp(2cx + ay)$ *is a first integral.*

(14) *If* $c = r = 0$ *and* $a - q = 0$, *then* $f_2 = \exp((2ax + by + 2p)/(y - x^2))$ *is an exponential factor and* $H = (2ax + by + 2p)/(y - x^2) + (2bp - a^2)t$ *is an invariant.*

(15) *If* $b = c = r = 0$ *and* $a - q = 0$, *then* $f_2 = \exp(2(ax + p)/(y - x^2))$ *is an exponential factor, and* $f_1^{\lambda_1} f_2^{\lambda_2}$ *is an integrating factor for* $\lambda_1 = -3$ *and* $\lambda_2 = -p/a^2$ *which gives the first integral* $H = (ax^2 - 2px - qy)/(y - x^2)\exp(-2p(ax + p)/(a^2(y - x^2)))$.

Proof. From proposition 4 we know that the cofactor of f_1 is $K_1 = ax + by + c$. To prove statements (1)–(3) we apply Theorem 3 with only f_1. Theorem 3(1) is applied to statements (4)–(9), (11)–(13) and (15). Under the assumptions of statement (5), the cofactor of f_2 is $K_2 = (ax + by + 2p + (a - 2q)a/b)/2$. Since $K_1 - 2K_2 = 0$, we obtain (5). For statement (4), the cofactor of f_2 is $K_2 = (a + 4q + |A|)x + by + (80cq^2 + 48bp^2 + 6a^2p + 9a^2c + 36bc^2 - 96bcp - 32pq^2 - 48acq + (9ac + 6ap - 8pq - 12cq)|A|)/(2(3a - 4q)(3a - 4q + 3|A|))$. Since $\lambda_1 K_1 + \lambda_2 K_2 = -$div for the values of λ_i's given in (4), we obtain (4). Similarly, statements (6)–(9), where the respective invariant straight lines have $qx + p/2$, p, $(q + a/4)x + by - a(a - 4q)/(8b)$, $4qx/3 + by + 4q^2/(9b)$ as cofactors, and statements (11), (12) and (15), where the respective exponential factors have $(a - 2q)(4ra - 4rq - c^2)/(2c)$, $a(c^2 - ar)/(4c)$ and a^2 as cofactors are proved. For statement (13), $\lambda_1 K_1 + \lambda_2 K_2 = 0$ with $K_2 = (2ar + 2pc - 4p^2)x + (-1/2a(2c - 4p) + ca)y + r(2c - 4p)$ as cofactor. Finally, Theorem 3(5) is applied to statements (10) and (14) with the cofactors $K_2 = (2q - a)(2r(q - a)/c + c/2)$, for statement (10), $K_2 = a^2 - 2bp$, for statement (14). □

Note that the conditions satisfied in Theorem 11(15) are the same as those of Theorem 11(14) plus the additional one $b = 0$. However, Theorem 11(15) is not a particular case of Theorem 11(14) because it concerns an integrating factor. Actually the conditions for Theorem 11(14) alone lead only to an invariant.

9. Quadratic systems having two invariant real parallel straight lines

We can write system (PL) in the form

$$\dot{x} = x^2 - 1 \qquad \dot{y} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2. \tag{9}$$

Proposition 12. *Let $f_1 = x + 1$ and $f_2 = x - 1$. If $a_{02} = 0$ in system (9), then $f_1^{\lambda_1} f_2^{\lambda_2}$ is an integrating factor for $\lambda_1 = (a_{01} - a_{11} - 2)/2$ and $\lambda_2 = -(a_{01} + a_{11} + 2)/2$.*

Proof. Let K_1 and K_2 be the cofactors of f_1 and f_2, respectively. Then $K_1 = x - 1$ and $K_2 = x + 1$. From Theorem 3(1) and since $\lambda_1 K_1 + \lambda_2 K_2$ is equal to minus the divergence of system (9), the lemma follows. □

Since we are interested in systems (9) which are integrable, by Proposition 12 we can assume that $a_{02} \neq 0$. Theorem 26 of [2] characterizes all systems (9) with $a_{02} \neq 0$ which are integrable having at least three algebraic curves of degree 1. Of course, systems (9) always have the two invariant straight lines $x + 1 = 0$ and $x - 1 = 0$.

Note that we have found exponential factors for system (PL) of the form $\exp(g(x, y)/h)$ with degree$(g) \leqslant 3$ and h is $1, x + 1, x - 1$ or $x^2 - 1$, but all the results obtained are included in Proposition 12.

10. Quadratic systems having two invariant complex parallel straight lines

We assume that $a \neq 0$ for the systems (CL), otherwise $x =$ constant is a first integral. Therefore, doing a rescaling of the time variable (if necessary) we can assume that $a = 2$, and consequently systems (CL) can be written in the form

$$\dot{x} = x^2 + 1 \qquad \dot{y} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2. \tag{10}$$

Proposition 13. *Let $f_1 = x + i$. If $a_{02} = 0$ for a system (10) of type (CL), then $f_1^{\lambda_1} \bar{f}_1^{\bar{\lambda}_1}$ is an integrating factor for $\lambda_1 = -\frac{1}{2}(2 + a_{11} + \mathrm{i}a_{01})$.*

Proof. The cofactor of f_1 is $K_1 = x - i$. From Theorem 3(1), since $\lambda_1 K_1 + \bar{\lambda}_1 \bar{K}_1$ is equal to minus the divergence of system (10), the lemma follows. □

Due to Proposition 13 in the rest of the section we assume that $a_{02} \neq 0$. Theorem 27 of [2] classifies all real quadratic systems of type (CL) which are integrable having at least three invariant algebraic curves of degree 1. Of course, systems (CL) always have two complex straight lines $x + i = 0$ and $x - i = 0$.

Note that we have found exponential factors for system (CL) of the form $\exp(g(x, y)/h)$ with degree$(g) \leqslant 3$ and $h = 1$ or $h = x^2 + 1$, but all the results obtained are included in Proposition 13.

11. Quadratic systems having an invariant real straight line

Suppose that a quadratic system has an invariant straight line. After a rotation and a translation we can assume that this invariant straight line is $x = 0$. Then, the system can be written as

$$\dot{x} = x(ax + by + c) \qquad \dot{y} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2. \tag{11}$$

If this system, which has the invariant straight line $x = 0$, has another invariant straight line or an invariant conic, then after an affine change of coordinates (if necessary) it can be written as one of the first eight normal forms of Proposition 12. Hence, we do not need to study the integrability of systems (DL) if they have some invariant algebraic curve of degree 1 or 2 different from $x = 0$.

In the next theorem, we study the integrability of systems (11) using the existence of the invariant straight line $x = 0$, and only looking for exponential factors of the form $\exp(g(x, y)) = 0$, $\exp(g(x, y)/x)$ and $\exp(g(x, y)/x^2) = 0$, with $\mathrm{degree}(g) \leqslant 3$. If in this study, there appear systems which are integrable due to the existence of some other invariant straight line or some invariant conic we omit them, because the integrability of such systems is known as mentioned above.

Table 5. Definition of the parameters appearing in theorem 14.

Name	Expression	Name	Expression	Name	Expression
a_1	$a - a_{11}$	a_2	$a + a_{11}$	a_3	$a - 2a_{11}$
a_4	$a + 2a_{11}$	a_5	$2a - a_{11}$	a_6	$2a + a_{11}$
a_7	$2a - 3a_{11}$	a_8	$3a - a_{11}$	b_1	$b - a_{02}$
b_2	$b + a_{02}$	b_3	$b - 2a_{02}$	b_4	$2b - a_{02}$
c_1	$c - a_{01}$	c_2	$c + a_{01}$	c_3	$c - 2a_{01}$
c_4	$2c - a_{01}$	d	$4a_{10}a_{02} - ac$	e_1	$a_{00}a - a_{10}c$
e_2	$a_{01}a - a_{10}b$	e_3	$a_{10}a - a_{20}c$	e_4	$a_{10}b - a_{11}c$

Theorem 14. *Let $f_1 = x = 0$, then if $a^2 + b^2 + c^2 \neq 0 = 0$, the following cases hold. (see Table 5):*

(1) *If $b_1 = 0$, $a_{00}b + c_1c = 0$ and $bc \neq 0$, then $f_2 = \exp((cy + a_{00})/x)$ is an exponential factor and the additional condition $a_{20}b + a_1a = 0$ gives the invariant $H = x^{a_1c} \exp\left(b(cy + a_{00})/x + \left(e_1b - a_1c^2\right) t\right)$.*

(2) *If $a_{02} = 0$, $a_{11}a - a_{20}b = 0$ and $ab \neq 0$, then $f_2 = \exp(-a_{20}x + ay)$ is an exponential factor and the additional condition $a_{00}b\left(a^2 + a_{20}b\right) - a_{01}a(ac - e_2) = 0$ leads, when $a_{01} \neq 0$, to the first integral $H = x^{-a_{01}^2 a}(a_{01}(ax + by) + a_{00}b)^{a(a_{01}c - a_{00}b)} \exp(a_{01}b(ay - a_{20}x))$. If the additional condition is $a_{10}ab - a_{01}a^2 - a_{20}bc = 0$, then $H = x^{a_{01}a} \exp(a_{20}bx - aby + a(a_{00}b - a_{01}c)t)$ is an invariant.*

(3) *If $a_{01} = a_{02} = a_2 = 0$, then $f_2 = \exp\left(-a_{20}x^2 + 2axy + by^2 - 2a_{10}x + 2cy\right)$ is an exponential factor, which gives the first integral $H = x^{2a_{00}} \exp\left(a_{20}x^2 - 2axy - by^2 + 2a_{10}x - 2cy\right)$.*

(4) *If $b_4 = 0$, $ba_{00} + c_4c = 0$, $c^2a_5 - be_1 = 0$ and $bc \neq 0$, then $f_2 = \exp\left((-2c(a_5x - by) + a_{00}b)/(2x^2)\right)$ is an exponential factor and $H = (2bcy + a_{00}b)/x^2 - 2a_5c/x - 2c\left(2a^2 - a_{11}a + a_{20}b\right) t$ is an invariant. Moreover, the additional condition $a_8 = 0$ gives the integrating factor $R = x^{-5} f_2^{\lambda_2}$ with $\lambda_2 = \left(a_{00}b - 2c^2\right)\big/\left(c^2\left(a^2 - a_{20}b\right)\right)$.*

(5) *If $b_2 = 0$, $a_2a + a_{02}a_{20} = 0$, $a^2c_2 + a_{02}e_3 = 0$ and $a_{02}ab \neq 0$, then $f_2 = \exp\left(x\left(a_{20}ax - 2a^2y + 2e_3\right)\right)$ is an exponential factor and the additional condition $a_{00}a^2 - e_3c = 0$ leads, when $a_{00}^2 + c^2 \neq 0$, to the first integral $H = x(a_{20}cx - 2acy + 2a_{00}a)$. Similarly, if the additional condition is $a^2c - a_{02}e_3 = 0$, then $R = xf_2^{\lambda_2}$ with $\lambda_2 = b(3a + a_{11})\big/\left(2a^2\left(c^2 + a_{00}b\right)\right)$ is an integrating factor.*

(6) *If $c = a_{00} = b_1 = 0$, then $f_2 = \exp((by + a_{01})/x)$ is an exponential factor, and the additional condition $a_{01}a_{20} + a_{10}a_1 = 0$ with $a_{01} \neq 0$, leads to the first integral $H = x^{a_{10}a_{02} - a_{01}a_{11}}(a_{10}x + a_{01}y)^{a_{01}a - a_{10}a_{02}} \exp((a_{02}y + a_{01})a_{01}/x)$.*

(7) *If* $a = a_{02} = a_{20} = 0$ *and* $b^2+a_{11}^2 \neq 0$, *then* $f_2 = \exp(-a_{11}x+by)$ *is an exponential factor, and the additional condition* $a_{01}a_{10} - a_{00}a_{11} = 0$ *leads, when* $a_{10}a_{11} \neq 0$, *to the first integral* $H = x^{a_{00}a_{11}^2}(a_{11}y + a_{10})^{a_{10}e_4}\exp(a_{11}a_{10}(a_{11}x - by))$. *If the additional condition is* $e_4 = 0$, *then one obtains the invariant* $H = x^{a_{01}}\exp(a_{11}x - by + (a_{00}b - a_{01}c)t)$.

(8) *If* $a_{00}a_{11} - a_{10}(2c - a_{01}) = 0$, $a_{00}a_{02} + 2c_1c_4 = 0$, $2a_{10}^2c_4 + a_{00}(a_{20}c_1 - a_{10}a) = 0$, $a_{00}b + 3c_1c_4 = 0$ *and* $a_{00}c_1b \neq 0$, *then* $H = \big(a_{10}(a_{00}a - 2a_{10}c_4)^2x^3 - a_{00}ac_1(a_{00}a - 2c_4a_{10})yx^2 + c_4c_1^2(-3c_4a_{10} + 2a_{00}a)xy^2 - c_4^2c_1^3y^3 - a_{00}c_1(a_{01}a_{00}a - 2c_4a_{10}c)xy - \frac{1}{2}a_{00}c_4(2c - 3a_{01})c_1^2y^2 - a_{00}^2c_1(-5a_{10}c + a_{00}a + 3a_{10}a_{01})x + a_{00}^2c_1^2cy + \frac{1}{2}a_{00}^3c_1^2\big)/x^2$ *is a first integral*.

(9) *If* $2b - 3a_{02} = 0$, $c(2c - 3a_{01}) + 3a_{00}b = 0$, $a_7c^2 - 3e_1b = 0$, $3a_{20}b + a_7a = 0$ *and* $bc \neq 0$, *then* $H = \big(-8a_7^3c^3x^3 - 24a_7^2bc^3x^2y - 24a_7b^2c^3xy^2 - 8b^3c^3y^3 - 72c^2b^2a_7a_{00}xy - 36a_{00}b^3c^2y^2 - 54a_{00}^2b^2c(a_7x + by) - 27a_{00}^3b^3\big)/x^2$ *is a first integral*.

(10) *If* $a_{00} = a_{11} = b_3 = c_1 = 0$, *then* $f_2 = \exp\big((a_{20}x^2 - axy - a_{02}y^2 - cy)/x\big)$ *is an exponential factor, and* $H = x^{a_{10}}\exp\big(((a_{20}x - ay)x - cy - a_{02}y^2)/x\big)$ *is a first integral*.

(11) *If* $a = a_{11} = b_3 = c_1 = 0$, *then* $f_2 = \exp\big((- 2a_{20}x^2 + by^2 + 2cy + 2a_{00})/x\big)$ *is an exponential factor, and* $H = x^{2a_{10}}\exp\big((2a_{20}x^2 - by^2 - 2cy - 2a_{00})/x\big)$ *is a first integral*.

(12) *If* $c = a_{01} = a_1 = b_1 = 0$, *then* $f_2 = \exp\big((by^2 + 2(ay + a_{10})x + a_{00})/x^2\big)$ *is an exponential factor, and* $H = x^{2a_{20}}\exp\big(-(by^2 + 2(ay + a_{10})x + a_{00})/x^2\big)$ *is a first integral*.

(13) *If* $a = a_{11} = b_1 = c_4 = 0$, *then* $\exp\big((by^2 + 2a_{10}x + 2cy + a_{00})/x^2\big)$ *is an exponential factor, and* $H = x^{-2a_{20}}\exp\big((by^2 + 2a_{10}x + 2cy + a_{00})/x^2\big)$ *is a first integral*.

(14) *If* $c = a_{01} = a_{11} = b_3 = 0$, *then* $f_2 = \exp\big((2(a_{20}x - ay)x - by^2 - 2a_{00})/x\big)$ *is an exponential factor, and* $H = x^{2a_{10}}\exp\big((2a_{20}x^2 - 2axy - by^2 - 2a_{00})/x\big)$ *is a first integral*.

(15) *If* $a_{00} = a_1 = b_1 = c_4 = 0$, *then* $f_2 = \exp\big((a_{02}y^2 + 2(ax + c)y + 2a_{10}x)/x^2\big)$ *is exponential factor, and* $H = x^{2a_{20}}\exp\big(-(a_{02}y^2 + 2(ax + c)y + 2a_{10}x)/x^2\big)$ *is a first integral*.

(16) *If* $a_{00} = a_{01} = a_{02} = 0$ *and* $ba_{20} - a_4a_6 = 0$, *then* $H = 2a_2\big(a_6a_4^2x^3 + b^3y^3\big) - 6ba_2(aa_4x + a_{11}by)xy + 3a_4(b(3a + 2a_{11})a_{10} - a_6a_4c)x^2 - 6b(aa_{10}b + a_{11}a_4c)xy - 3b^2(a_{10}b - c(3a_{11} + 2a))y^2 + 6b(a_{10}b - ca_4)(a_{10}x - cy)$ *is a first integral*.

(17) *If* $a_{00} = c_1 = b_3 = 0$, $a_3a + 2a_{20}b = 0$ *and* $a_{02}ab \neq 0$, *then* $f_2 = \exp\big((4a_{20}^2a_{02}x^2 - 4a_{02}a_{20}axy + a_{02}a^2y^2 + a^2cy)/x\big)$ *is an exponential factor, and if moreover,* $4a_{02}a_{20}c - ad = 0$, *then* $H = \big(d^2x^2 - 4a_{02}cdxy + 4a_{02}^2c^2y^2 + 4c^3a_{02}y\big)/x - 4c^3a_{10}a_{02}t$ *is an invariant. If the additional condition is* $c = 0$, *then* $R = x^{-5/2}f_2^{1/(2a_{10}a^2)}$ *is an integrating factor. Finally, if the additional condition is* $4a_{02}a_{20} + a^2 = 0$, *then* $R = f_2^{-2/(a_{10}a^2)}$ *is an integrating factor*.

(18) *If* $a = a_{20} = b_3 = c_1 = 0$, *then* $f_2 = \exp\big((4a_{11}^2x^2 - 4a_{11}bxy + b^2y^2 + 2bcy + 2a_{00}b)/x\big)$ *is an exponential factor. If* $b \neq 0$ *and if moreover,* $2a_{00}b - c^2 = 0$, *then* $R = x^{-5/2}f_2^{1/(4a_{10}b - 4a_{11}c)}$ *is an integrating factor*.

(19) *If* $a_{00} = b_3 = c_3 = 0$ *and* $a_3a + 4a_{02}a_{20} = 0$, *then* $f_2 = \exp\big((2a_{20}x - ay)^2/x\big)$ *is an exponential factor. If* $a_{02}ab \neq 0$, *then the additional condition* $4a_{20}a_{02} + a^2 = 0$ *leads to the invariant* $H = x^{4e_3a_{20}}\exp\big(a(-2a_{20}x + ay)^2/x - 4a_{20}e_3ct\big)$ *and to the integrating factor* $R = x^{-3/2}f_2^{a/(8a_{20}e_3)}$. *Finally, if the additional condition is* $c = 0$, *then* $R = x^{-5/2}f_2^{a_{02}/(2a_{10}a^2)}$ *is an integrating factor*.

(20) *If* $c = a_{00} = a_{11} = b_3 = 0$, *then* $f_2 = \exp\big((-2a_{20}bx^2 + 2abxy + b^2y^2 + 4a_{01}by + 4a_{01}^2)/x\big)$ *is an exponential factor. If* $b \neq 0$ *and if moreover,* $2a_{20}b + a^2 = 0$, *then* $H = x^{-2a_{10}b + 2a_{01}a}\exp\big(((ax + by)^2 + 4a_{01}(a_{01} + by))/x + 4a_{01}(a_{01}a - a_{10}b)t\big)$ *is an invariant and* $R = x^{-3/2}f_2^{1/(4e_2)}$ *is an integrating factor*.

(21) *If* $c = a_{00} = b_4 = e_2 = 0$ *and* $a \neq 0$, *then* $f_2 = \exp\big((2a(-a_5x + by) + a_{10}b)/x^2\big)$ *is an exponential factor, and* $H = (2a(a_5x - by) - a_{10}b)/x^2 + 2a(a_{20}b + a_5a)t$ *is an invariant. If* $b \neq 0$ *and if moreover,* $a_8 = 0$, *then* $H = a\big(a^2 - a_{20}b\big)x^2 - \big(a_{10}abx - a_{10}b^2y\big)^{2a^2(a^2-a_{20}b)} \exp\big(a_{10}b\big(2a^2x + 2aby + a_{10}b\big)/x^2\big)/x^{4a^2(a^2-a_{20}b)}$ *is a first integral. Finally, if the additional condition is* $a_5a + a_{20}b = 0$, *then* $H = (2a(a_5x - by) - a_{10}b)/x^2$ *is a first integral.*

(22) *If* $a = c = a_{00} = a_{10} = b_4 = 0$, *then* $f_2 = \exp\big((2a_{11}x + 2by + a_{01})/x^2\big)$ *is an exponential factor, and* $H = (2a_{11}x + 2by + a_{01})/x^2 - 2a_{20}bt$ *is an invariant. If moreover,* $a_{11} = 0$, *then* $H = \big(a_{20}x^2 + a_{01}y\big)\exp\big(-a_{01}(2by + a_{01})/\big(2a_{20}bx^2\big)\big)/x^2$ *is a first integral.*

Proof. The cofactor of f_1 is $ax + by + c$ and the cofactors of f_2 are $a_{20}cx - ca_1y - e_1$, $e_3x + a_{01}ay + a_{00}a$, $2a_{00}(ax + by + c)$, $c(a_5a + a_{20}b)$, $-2a\big(a_2c^2 + e_1b\big)x/b$, $b(a_{20}x - a_1y) + a_{10}b - a_{01}a$ and $e_4x + b(a_{01}y + a_{00})$, respectively, for statements (1)–(7); $-(ax + 2a_{02}y + c)a_{10}$, $2a_{10}(by + c)$, $2a_{20}(ax + by)$, $2a_{20}(by + c)$, $-2a_{10}(ax + by)$ and $2a_{20}(ax + a_{02}y + c)$, respectively, for statements (10)–(15), $\big(a^2c - 4a_{02}e_3\big)(2a_{20}x - ay)/2 + a_{10}a^2c$, $2e_5(by - 2a_{11}x) + 2b(a_{10}c - 2a_{11}a_{00})$, $2e_3(-2a_{20}x + ay)$, $2b(2a_{01}a_{20} + a_{10}a)x - 2(a_{01}a - a_{10}b)(by + 2a_{01})$, $2a(a_5a + a_{20}b)$ and $2a_{20}b$, respectively, for statements (17)–(22) and null for statemnts (8), (9) and (16). The different statements are proved by applying Theorem 3 with f_1 and f_2. □

12. Conclusion

All the differential systems considered are particular subclasses of real quadratic systems having an invariant conic as solution. These subclasses are obtained imposing suitable conditions on the parameters of general quadratic systems. It turns out that in these subclasses of systems, the existence of one invariant conic without any additional condition among the parameters of the differential system has been the key point for the search for first integrals, integrating factors and invariants. The use of both invariant algebraic curves and exponential factors have completed the package of results, as appears clearly in the Lotka–Volterra system. This paper completes the study of Darboux integrability of the quadratic differential systems having an invariant conic (see [2]). Here we restricted our interest to real systems. Without any serious difficulty the extension to complex systems can be done. It is also possible to investigate the Darboux integrability of these subclasses of quadratic systems looking for invariant algebraic curves and exponential factors given by polynomials of higher degree.

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Appendix. Definition of c_1 and c_2 in Theorem 11(4)

Here we give the definitions of c_1 and c_2 as follows: $c_1 = 32q^3a - (16pq + 24cq)ab - 3a^4 - 4|A| (-(6p - 3c)ab - 4q(c - 2p)b - 3a^3 + 10qa^2 - 8q^2a) + q(8pq - 20cq)b + 48b^2ra + 22a^3q - 48a^2q^2 + 4(-16rq - 24pc + 9c^2 + 12p^2)b^2 + 9(2p - c)ba^2 = 0$, $c_2 = 2916a^7q - 21888a^6q^2 + 88704a^5q^3 - 209408a^4q^4 + 287744a^3q^5 - 212992q^6a^2 + 65536q^7a - 162a^8 + 65536abpq^5 - 128b^3(2p - c)(212p^2 - 372pc + 153c^2)q^2 - 32768b(2p - c)q^6 + 192(164p^2 - 252pc + 81c^2)\, q(2p - c)ab^3 + 54b(50p - 33c)a^6 -$

$256\,q^3(706p - 603\,c)a^3b + 4096\,b^2(2p - c)(14p - 11c)q^4 - 144q(194p - 135c)a^5b + 48q(20\,p^2 - 516pc + 297\,c^2)b^2a^3 - 256\,q^3(189\,c^2 + 708\,p^2 - 772pc)\,ab^2 + 5\,632pq^2(14p - 9c)a^2b^2 + 1536b(62p - 79c)q^4a^2 - 216(2p - c)(44p^2 - 60pc + 15c^2)a^2b^3 - 144(16p - 9c)(2p - 3c)b^2a^4 + 26\,880q^2(4p - 3c)ba^4 + 288(2p - c)^2(2p - 3c)^2b^4 + (3a - 4q)(6\,144q^6 - 15\,104q^5a + 1476q^2a^4 - 128q^4(62p - 19c)b + 96p(2p - c)(2p - 3c)b^3 - 6\,640q^3a^3 + 14\,400q^4a^2 - 126qa^5 + 16q^2(34p + 21c)(2p - c)b^2 + 3(220p^2 - 27c^2 - 120pc)b^2a^2 - 9ba^4\ (46p - 9c) - 24q(6p + c)(14p - 9c)b^2a + 128q^3(127p - 40c)ba - 88q^2(134p - 39c)ba^2 + 12q\ (304p - 75c)ba^3)|A| + 3(3a - 4q)(336q^3a - 128q^4 - 18a^4 - 8q^2(6p - 7c)b + 126a^3q - 316a^2q^2 + 9(2p - c)(2p - 3c)b^2 + 3(2p - 3c)ba^2 + 4q(4p - 3c)ab)|A|^3 = 0.$

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